

NEW YORK UNIVERSITY
COURANT INSTITUTE — LIBRARY
4 Washington Place, New York 3, N. Y.

IMM-NYU 318
FEBRUARY 1964



NEW YORK UNIVERSITY
COURANT INSTITUTE OF
MATHEMATICAL SCIENCES

Automorphic Forms and Poincaré Series for Infinitely Generated Fuchsian Groups

LIPMAN BERS

PREPARED UNDER
CONTRACT NO. DA-ARO-(D)-31-124-G156
WITH THE
UNITED STATES ARMY RESEARCH OFFICE

IMM-318

C.1

New York University
Courant Institute of Mathematical Sciences

AUTOMORPHIC FORMS AND POINCARÉ SERIES FOR
~~T~~INFINITELY GENERATED FUCHSIAN GROUPS

Lipman Bers

This report represents results obtained at the Courant Institute of Mathematical Sciences, New York University, with the United States Army Research Office, Contract No. DA-ARO-(D)-31-124-G156. Reproduction in whole or in part is permitted for any purpose of the United States Government.

§1. Statement of results.

1. Let D be a simply connected domain in the extended complex plane with at least two boundary points, and G a discrete group of conformal self-mappings $z \rightarrow A(z)$ of D . If D is the upper half-plane U or the unit disc Δ the elements $A \in G$ are Möbius transformations and G is a Fuchsian group (or a Fuchsoid group in Poincaré's original terminology since we do not assume G to be finitely generated). While this can be always achieved by a conformal mapping, there are some advantages in considering the seemingly more general case of an arbitrary D .

Let

$$(1) \quad q \geq 2$$

be a fixed integer. An automorphic form of weight $(-2q)$ is a holomorphic solution of the functional equation

$$(2) \quad \phi(A(z))A'(z)^q = \phi(z) \quad \text{for } z \in D, \quad A \in G.$$

We require in addition that

$$(3) \quad \phi(z) = O(|z|^{-2q}), \quad z \rightarrow \infty \quad \text{if } \infty \in D.$$

Let $\lambda_D(z)|dz|$ denote the Poincaré metric in D . The automorphic forms with

$$(4) \quad \|\phi\|_{A_q(D,G)} = \iint_{D/G} \lambda_D(z)^{2-q} |\phi(z)| dx dy < \infty$$

form the Banach space $A_q(D,G)$ of integrable forms. The automorphic forms with

1940年10月10日

1940年10月10日，星期日，晴，风和日丽。

上午九时，由上海乘火车赴南京，约十时抵京。

下午二时，由南京乘火车赴徐州，约四时抵徐。

下午六时，由徐州乘火车赴蚌埠，约八时抵蚌。

下午九时，由蚌埠乘火车赴合肥，约十一时抵合。

下午二时，由合肥乘火车赴六安，约四时抵六。

下午六时，由六安乘火车赴霍山，约八时抵霍。

下午九时，由霍山乘火车赴舒城，约十一时抵舒。

下午二时，由舒城乘火车赴庐江，约四时抵庐。

20

1940年10月11日

上午九时，由庐江乘火车赴无为，约十一时抵无。

下午二时，由无为乘火车赴和县，约四时抵和。

上午九时，由和县乘火车赴含山，约十一时抵含。

下午二时，由含山乘火车赴当涂，约四时抵当。

上午九时，由当涂乘火车赴繁昌，约十一时抵繁。

上午九时，由繁昌乘火车赴南陵，约十一时抵南。

下午二时，由南陵乘火车赴泾县，约四时抵泾。

上午九时，由泾县乘火车赴绩溪，约十一时抵绩。
(A)

上午九时，由绩溪乘火车赴旌德，约十一时抵旌。

下午二时，由旌德乘火车赴泾县，约四时抵泾。

$$(5) \quad \|\phi\|_{B_q(D,G)} = \text{ess. sup } \lambda_D(z)^{-q} |\phi(z)|$$

form the Banach space $B_q(D,G)$ of bounded forms. For $\phi \in A_q(D,G)$, $\psi \in B_q(D,G)$ the Petersson scalar product is defined by

$$(6) \quad (\phi, \psi)_{q,G} = \iint_{D/G} \lambda_D(z)^{2-2q} \phi(z) \overline{\psi(z)} dx dy .$$

In (4) and (6) the integration is performed over an arbitrary fundamental region ω of G in D . This means that $\omega \subset D$ is measurable, $\text{mes Int } (\omega) = \text{mes } \omega$, $A(z_1) = z_2$ for $z_1, z_2 \in \text{Int } (\omega)$ and $\text{id} \neq A \in G$, and $D = \bigcup_{A \in G} A(\omega)$.

If $G = \{\text{id}\}$, we write $A_q(D)$, $B_q(D)$ and $(\phi, \psi)_q$ instead of $A_q(D,G)$, $B_q(D,G)$ and $(\phi, \psi)_{q,G}$. Clearly $A_q(D,G) \cap A_q(D) = \{0\}$ unless G is finite, while $B_q(D,G)$ is always a closed linear subspace of $B_q(D)$.

2. If $D = U$ and G has a fundamental region of finite non-Euclidean area,

$$(7) \quad \iint_{D/G} \lambda_D(z)^2 dx dy < \infty ,$$

then $A_q(D,G) = B_q(D,G)$ is the finite dimensional space of so-called cuspidal forms. In the general case we have

Theorem 1. The Petersson product establishes an anti-isomorphism between $B_q(D,G)$ and the dual space to $A_q(D,G)$.

It is trivial that, for a fixed $\psi \in B_q(D,G)$,

$$\ell(\phi) = (\phi, \psi)_{q,G}$$

THE JOURNAL OF THE AMERICAN MEDICAL ASSOCIATION
PUBLISHED WEEKLY
CHICAGO, ILL., MAY 1, 1919
Vol. 27, No. 19

THE JOURNAL OF THE AMERICAN MEDICAL ASSOCIATION
PUBLISHED WEEKLY
CHICAGO, ILL., MAY 1, 1919
Vol. 27, No. 19

THE JOURNAL OF THE AMERICAN MEDICAL ASSOCIATION
PUBLISHED WEEKLY
CHICAGO, ILL., MAY 1, 1919
Vol. 27, No. 19

THE JOURNAL OF THE AMERICAN MEDICAL ASSOCIATION
PUBLISHED WEEKLY
CHICAGO, ILL., MAY 1, 1919
Vol. 27, No. 19

THE JOURNAL OF THE AMERICAN MEDICAL ASSOCIATION
PUBLISHED WEEKLY
CHICAGO, ILL., MAY 1, 1919
Vol. 27, No. 19

THE JOURNAL OF THE AMERICAN MEDICAL ASSOCIATION
PUBLISHED WEEKLY
CHICAGO, ILL., MAY 1, 1919
Vol. 27, No. 19

THE JOURNAL OF THE AMERICAN MEDICAL ASSOCIATION
PUBLISHED WEEKLY
CHICAGO, ILL., MAY 1, 1919
Vol. 27, No. 19

THE JOURNAL OF THE AMERICAN MEDICAL ASSOCIATION
PUBLISHED WEEKLY
CHICAGO, ILL., MAY 1, 1919
Vol. 27, No. 19

THE JOURNAL OF THE AMERICAN MEDICAL ASSOCIATION
PUBLISHED WEEKLY
CHICAGO, ILL., MAY 1, 1919
Vol. 27, No. 19

THE JOURNAL OF THE AMERICAN MEDICAL ASSOCIATION
PUBLISHED WEEKLY
CHICAGO, ILL., MAY 1, 1919
Vol. 27, No. 19

THE JOURNAL OF THE AMERICAN MEDICAL ASSOCIATION
PUBLISHED WEEKLY
CHICAGO, ILL., MAY 1, 1919
Vol. 27, No. 19

THE JOURNAL OF THE AMERICAN MEDICAL ASSOCIATION
PUBLISHED WEEKLY
CHICAGO, ILL., MAY 1, 1919
Vol. 27, No. 19

THE JOURNAL OF THE AMERICAN MEDICAL ASSOCIATION
PUBLISHED WEEKLY
CHICAGO, ILL., MAY 1, 1919
Vol. 27, No. 19

THE JOURNAL OF THE AMERICAN MEDICAL ASSOCIATION
PUBLISHED WEEKLY
CHICAGO, ILL., MAY 1, 1919
Vol. 27, No. 19

is a continuous linear functional on $A_q(D, G)$, of norm

$\|\ell\| \leq \|\psi\|_{A_q(D, G)}$. To prove Theorem 1 we will have to show that every ℓ can be so represented and that $\psi = 0$ whenever $(\phi, \psi)_{q, G} = 0$ for all $\phi \in A_q(D, G)$.

3. Let $\overline{\Phi}(z)$, $z \in D$, be a holomorphic function. We say that $(\kappa)_{q, G}\overline{\Phi}$ exists if

$$(8) \quad ((\kappa)_{q, G}\overline{\Phi})(z) = \sum_{A \in G} \overline{\Phi}(A(z)) A'(z)^q$$

where the Poincaré series to the right converges absolutely and uniformly on compact subsets of D . In this case $(\kappa)_{q, G}\overline{\Phi}$ is an automorphic form of weight $(-2q)$. It is known that if (7) holds, every cusp form is a Poincaré series. In the general case we have

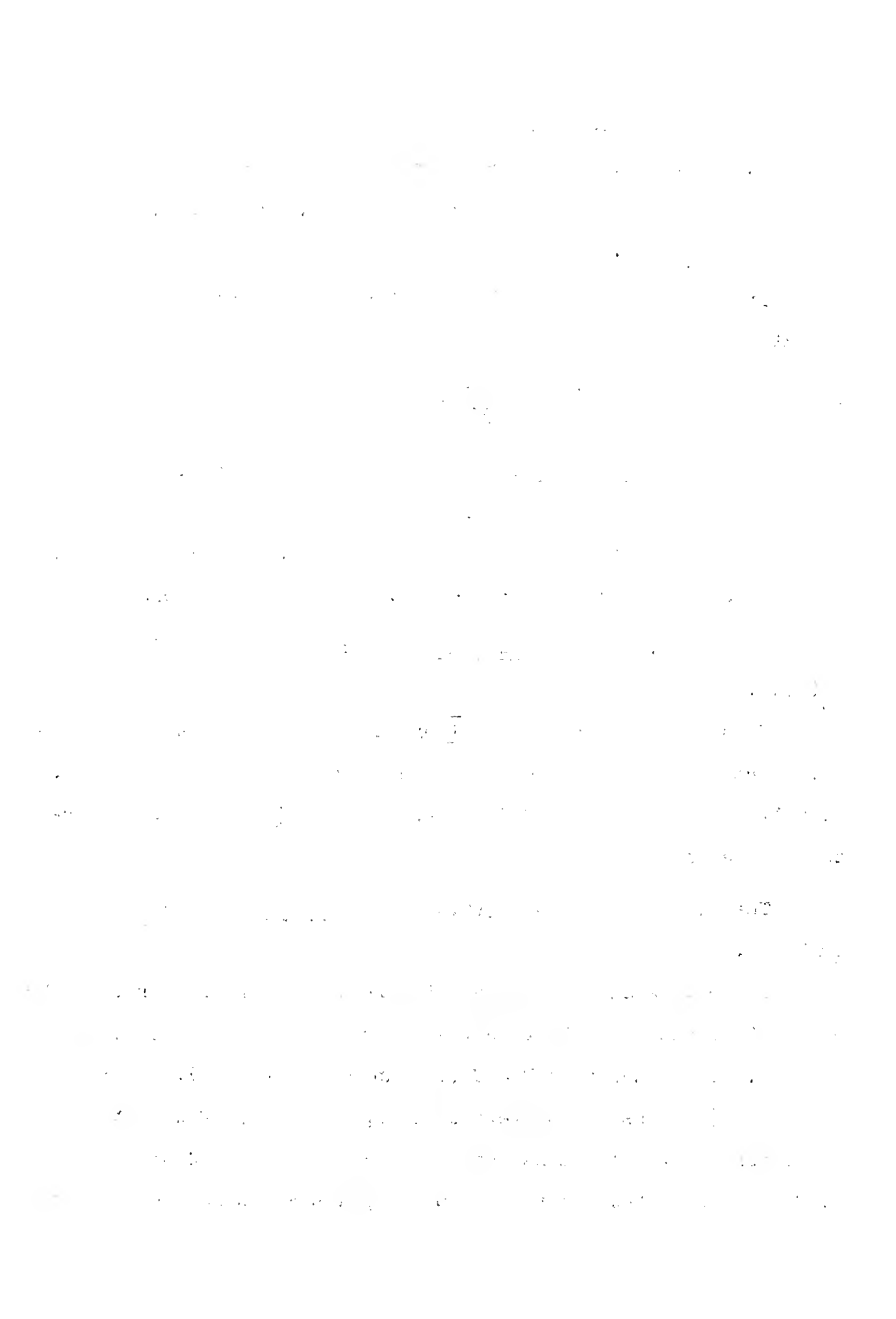
Theorem 2. $(\kappa)_{q, G}$ is a continuous mapping of $A_q(D)$ onto $A_q(D, G)$.

Thus, for $\overline{\Phi} \in A_q(D)$, $(\kappa)_{q, G}\overline{\Phi}$ exists and every $\phi \in A_q(D, G)$ is of this form. If $\overline{\Phi} \in B_q(D)$, however, the series in (8) may diverge. It will certainly do so if G is infinite and $\overline{\Phi} \in B_q(D, G)$. Nevertheless we have

Theorem 3. Every $\psi \in B_q(D, G)$ is of the form $\psi = (\kappa)_{q, G}\overline{\Psi}$, $\overline{\Psi} \in B_q(D)$.

Theorems 2 and 3 supercede the results of [2]. For the sake of completeness we shall repeat some arguments from that paper.

4. Assume now that $D = U$ (the upper half-plane). Following Eichler [5] we assign to every automorphic form ϕ of weight $(-2q)$ an element of the 1-dimensional cohomology group of G with coefficients in the additive group of polynomials in one variable



of degree at most $2q-2$, the Eichler class of ϕ (cf. 20 below). It is known that under hypothesis (7) a cusp form is uniquely determined by its Eichler class.

Theorem 4. If $D = U$, G is of the first kind, and the Eichler class of $\phi \in B_q(U, G)$ vanishes, then $\phi = 0$.

We recall that G is said to be of the first or second kind according to whether the whole real axis is or is not contained in the closure $\bigwedge(G)$ of the set of fixed points of elements of G . If G is of the second kind, $\bigwedge(G)$ is either a perfect nowhere dense set or contains less than three points. In the latter case G is called elementary.

Theorem 5. Let $D = U$ and let G be a non-elementary group of the second kind. The Eichler class of $\phi \in B_q(U, G)$ vanishes if and only if ϕ is orthogonal to all forms $\bigotimes_{q, G} \overline{\Phi}$, where $\overline{\Phi} \in A_q(U)$ is a rational function with poles in $\bigwedge(G)$.

If G is of the second kind, we denote by $A_2^\#(U, G)$ the set of those $\phi \in A_2(U, G)$ which are continuous and real on the real axis off $\bigwedge(G)$.

Theorem 6. Let G be as in Theorem 5. The Eichler class of $\phi \in B_2(U, G)$ vanishes if and only if ϕ is orthogonal to $A_2^\#(U, G)$.

In Theorems 5 and 6 orthogonality is meant in the sense of the Petersson product. Theorem 4 and suitably modified forms of Theorems 5 and 6 hold also for $D = \Delta$ (the unit disc).

5. Let D_G denote the set D from which the fixed points of elements of G distinct from the identity have been removed. The set D/G has a canonical conformal structure defined by the

requirement that the projection $D \rightarrow D/G$ be a holomorphic mapping. Thus D/G and $D_G/G \subset D/G$ are Riemann surfaces. Let π_1 denote the fundamental group.

Theorem 7. G is finitely generated if and only if $\pi_1(D_G/G)$ is.

The statement is trivial if G is a fixed point free in D , (for then $D = D_G$ and since D is simply connected G is isomorphic to $\pi_1(D/G)$). It is "well known" in all cases. But a direct proof has the advantage of enabling one to base the theory of finitely generated Fuchsian groups on uniformization theory to which an easy access via quasi-conformal mappings is now available (cf. [2]). Recently Ahlfors [1] extended Theorem 7 to Kleinian groups. Our proof of Theorem 7 is based on Theorems 4 and 6. We remark that while the proof of Theorem 4 is almost trivial the reduction of Theorem 6 to Theorem 5 depends on a device employed by Ahlfors.

82. Preliminaries.

6. Let $f(z)$ be a conformal mapping of D . The Poincaré metric has the property that

$$(9) \quad \lambda_D(z)|dz| \text{ is a conformal invariant.}$$

This means that $\lambda_{f(D)}(f(z))|f'(z)| = \lambda_D(z)$.

For every $A \in G$ set $\hat{A} = f \circ A \circ f^{-1}$. These \hat{A} 's form a discrete group \hat{G} of conformal self-mappings of $f(D)$. For every function $\phi(\zeta)$, $\zeta \in f(D)$, set $(f^*\phi)(z) = \phi(f(z))f'(z)^q$. Noting condition (3) we verify that f^* is an isometric linear mapping of $A_q(f(D), \hat{G})$

onto $A_q(D, G)$ and of $B_q(f(D), \hat{G})$ onto $B_q(D, G)$ which preserves the Petersson product:

$$(f^*\phi, f^*\psi)_{q, \hat{G}} = (\phi, \psi)_{q, G}.$$

One also verifies that

$$(\kappa)_{q, G} f^* \bar{\Phi} = f^* (\kappa)_{q, \hat{G}} \bar{\Phi}$$

where the existence of one side implies that of the other. Hence it suffices to prove Theorems 1-3 for some fixed domain D .

7. We have that

$$(10) \quad A_q(D) \subset B_q(D),$$

this injection being a continuous mapping.

It suffices to prove this for $D = U$ (cf. 6) and since

$$(11) \quad \lambda_U(z) = |z - \bar{z}|^{-1}$$

the assertion follows by a standard estimate:

$$\begin{aligned} |\phi(z)| &\leq \frac{4}{\pi y^2} \iint_{|\zeta - z| < y/2} |\phi(\zeta)| d\xi d\eta \\ &\leq \frac{4}{\pi y^2} \iint_{|\zeta - z| < y/2} \left(\frac{2\eta}{3y}\right)^{q-2} |\phi(\zeta)| d\xi d\eta \leq \frac{2^q}{3^{q-2} \pi y^2} \iint_{\eta > 0} \eta^{q-2} |\phi(\zeta)| d\xi d\eta \end{aligned}$$

so that $\|\phi\|_{B_q(U)} \leq 2^q 3^{2-q} \pi^{-1} \|\phi\|_{A_q(U)}$.

8. The Bergman kernel function $k_D(z, \zeta)$, $z \in D$, $\zeta \in D$ may be defined by the requirements

$$(12) \quad k_U(z, \zeta) = -1/\pi(z - \bar{\zeta})^2,$$

$$(13) \quad k_D(z, \zeta) dz d\bar{\zeta} \text{ is a conformal invariant,}$$

which means that $k_{f(D)}(f(z), f(\zeta)) \overline{f'(z) f'(\zeta)} = k_D(z, \zeta)$ for every conformal mapping f of D . The kernel $k_D(z, \zeta)$ is a holomorphic function of z and $\bar{\zeta}$ and

$$(14) \quad k_D(\zeta, z) = \overline{k_D(z, \zeta)}, \quad \pi k_D(z, z) = \lambda_D(z)^2.$$

Also,

$$(15) \quad \iint_D \lambda_D(\zeta)^{2-q} |k_D(z, \zeta)|^q d\xi d\eta = C_q \lambda_D(z)^q$$

where C_q is a constant. In view of (9) and (12) it suffices to verify this for $D = U$ in which case (15) follows from the identity

$$\iint_{\eta > 0} \frac{y^q \eta^{q-2} d\xi d\eta}{|x + iy - \xi + i\eta|^{2q}} = \int_{-\infty}^{+\infty} \frac{d\xi}{(1 + \xi^2)^q} \int_0^{+\infty} \frac{\eta^{q-2} d\eta}{(1 + \eta)^{2q-1}}$$

for $y > 0$.

From now on we omit the subscript D . The following re-producing formula holds (as it does also in bounded homogeneous domains in several variables, cf. Selberg [6]):

$$(16) \quad \phi(z) = c_q \iint_D \lambda(\zeta)^{2-2q} k(z, \zeta)^q \phi(\zeta) d\xi d\eta$$

for $\phi \in B_q(D)$, where

$$(16') \quad c_q = (2q-1)\pi^{q-1}.$$

It suffices to verify this for $D = \Delta$. Since

$$(17) \quad k_{\Delta}(z, \zeta) = 1/\pi(1 - z\bar{\zeta})^2, \quad \lambda_{\Delta}(z) = (1 - |z|^2)^{-1}$$

and

$$(2q - 1) \iint_{|\zeta| < 1} \frac{(1 - |\zeta|^2)^{2q-2} \zeta^m d\xi d\eta}{(1 - z\bar{\zeta})^{2q}} = \pi z^m, \quad m = 0, 1, \dots$$

the assertion follows.

9. Let $L_1(D)$ and $L_{\infty}(D)$ denote the usual complex Banach spaces of (equivalence classes of) integrable and bounded measurable functions in D . For $\mu \in L_1(D)$ set

$$(18) \quad (\alpha_q \mu)(z) = c_q \iint_D \lambda(\zeta)^{-q} k(z, \zeta)^q \mu(\zeta) d\xi d\eta,$$

and for $v \in L_{\infty}(D)$ set

$$(19) \quad (\beta_q v)(z) = c_q \iint_D \lambda(\zeta)^{2-q} k(z, \zeta)^q v(\zeta) d\xi d\eta.$$

By (15) the mappings α_q and β_q are continuous linear mappings of $L_1(D)$ and $L_{\infty}(D)$ into $A_q(D)$ and $B_q(D)$, respectively. These mappings are onto, since

$$(20) \quad \alpha_q(\lambda^{2-q} \phi) = \phi \quad \text{for} \quad \phi \in A_q(D),$$

$$(21) \quad \beta_q(\lambda^{-q} \phi) = \phi \quad \text{for} \quad \phi \in B_q(D),$$

by (10) and (16). Also

$$(22) \quad (\alpha_q \mu, \psi)_q = \iint_D \mu(z) \lambda(z)^{-q} \overline{\psi(z)} dx dy \quad \text{for} \quad \psi \in B_q(D),$$

and

$$(23) \quad (\phi, \beta_q v)_q = \iint_D \lambda(z)^{2-q} \phi(z) \overline{v(z)} dx dy \quad \text{for } \phi \in A_q(D) .$$

The proof involves merely substitution into the definition (6) for $G = \{\text{id}\}$, a change of order of integration, and an application of (16).

10. Let ℓ be a continuous linear functional on $A_q(D)$. By the theorems of Hahn-Banach and F. Riesz there is a $v \in L_\infty(D)$ such that

$$\ell(\phi) = \iint_D \lambda_D(z)^{2-q} \phi(z) \overline{v(z)} dx dy .$$

Hence, by (23) we have that $\ell(\phi) = (\phi, \psi)_q$ where $\psi = \beta_q v$. Next, let $\psi \in B_q(D)$ be such that $(\phi, \psi)_q = 0$ for all $\phi \in A_q(D)$. Noting (22) we conclude that

$$\iint_D \lambda(z)^{-q} \overline{\psi(z)} \mu(z) dx dy = 0$$

for all $\mu \in L_1(D)$. Hence $\psi \equiv 0$. Thus we have proved Theorem 1 for the case $G = \{\text{id}\}$.

§3. Poincaré series and duality.

11. We prove now that $\textcircled{\kappa}_q = \textcircled{\eta}_{q,G}$ is a continuous mapping of $A_q(D)$ into $A_q(D, G)$.

Let $\overline{\phi} \in A_q(D)$ and let ω denote a fundamental region of G in D . Then



$$\begin{aligned}
& \iint_{\omega} \lambda(z)^{2-q} \left| \sum_{A \in G} \overline{\Phi(A(z))} A'(z)^q \right| dx dy \\
& \leq \sum_{A \in G} \iint_{\omega} \lambda(z)^{2-q} |\overline{\Phi(A(z))} A'(z)^q| dx dy \\
& = \sum_{A \in G} \iint_{\omega} \lambda(A(z))^{2-q} |\overline{\Phi(A(z))}| |A'(z)|^2 dx dy \\
& = \sum_{A \in G} \iint_{A(\omega)} \lambda(z)^{2-q} |\overline{\Phi(z)}| dx dy = \|\overline{\Phi}\|_{A_q(D)} .
\end{aligned}$$

This implies the absolute and uniform convergence of the series (8) in every compact subset of a fundamental region and hence on every compact subset of D , as well as the inequality

$$\|\oplus_q \overline{\Phi}\|_{A_q(D,G)} \leq \|\overline{\Phi}\|_{A_q(D)} .$$

(Here we used two well known facts: L_1 convergence of holomorphic functions implies normal convergence. If $D_0 \subset \subset D$ there is an $\omega_0 \subset \subset \omega$ and a finite sequence $\{A_1, \dots, A_n\} \subset G$ such that $D_0 \subset A_1(\omega_0) \cup \dots \cup A_n(\omega_0)$.)

12. Let ℓ be a continuous linear functional on $A_q(D, G)$.

Let ω be a fundamental region. Then, by Hahn-Banach and F. Riesz,

$$(24) \quad \ell(\phi) = \iint_{\omega} \lambda(z)^{2-q} \phi(z) v(z) dx dy$$

with a bounded measurable $v(z)$. We extend v over the whole of D by the relation

$$(25) \quad v(A(z)) \overline{(A'(z)/A'(z))}^{q/2} = v(z) \quad \text{for } A \in G$$

(where $(\overline{A'}/A')^{q/2} = |A'|^q (A')^{-q}$). For $\overline{\Phi} \in A_q(D)$ we have

$$(26) \quad \ell(\bigoplus_q \bar{\Phi}) = \iint_D \lambda(z)^{2-q} \bar{\Phi}(z) v(z) dx dy$$

as follows from the identity

$$\begin{aligned} \iint_{\omega} \lambda(z)^{2-q} \sum \bar{\Phi}(A(z)) A'(z)^q v(z) dx dy \\ = \sum_{A \in G} \iint_{A(\omega)} \lambda(z)^{2-q} \bar{\Phi}(z) v(z) dx dy . \end{aligned}$$

Using this we shall show that

$$(27) \quad \ell(\bigoplus_q \bar{\Phi}) = 0 \quad \text{for all } \bar{\Phi} \in A_q(D)$$

implies that

$$(28) \quad \ell(\phi) = 0 \quad \text{for all } \phi \in A_q(D, G) ,$$

which means that

$$(29) \quad \bigoplus_q A_q(D) \text{ is dense in } A_q(D, G) .$$

During this proof we assume that $D = \Delta$ (the unit disc) and 0 is not a fixed point of any element of G distinct from the identity. This assumption involves no loss of generality.

13. For v satisfying (25) for $D = \Delta$ and such that the corresponding functional ℓ vanishes on $\bigoplus_q A_q(\Delta)$ set

$$(30) \quad h(z) = - \frac{1}{\pi} \iint_{|\xi| < 1} \frac{(1 - |\xi|^2)^{q-2} v(\xi) d\xi d\eta}{\xi - z}$$

and, for some fixed θ , $0 < \theta < 2\pi$,

$$(31) \quad \tilde{h}(z) = - \frac{(1 - ze^{-i\theta})^{q-2}}{\pi} \iint_{|\zeta| < 1} \frac{(1 - |\zeta|^2)^{q-2} v(\zeta) d\xi d\eta}{(1 - \zeta e^{-i\theta})^{q-2} (\zeta - z)} .$$

For a fixed z such that $|z| \geq 1$ the functions

$$\Omega(\zeta) = - \frac{1}{\pi} \frac{1}{\zeta - z} , \quad \tilde{\Omega}(\zeta) = - \frac{1}{\pi} \frac{1}{(1 - \zeta e^{-i\theta})^{q-2} (\zeta - z)}$$

belong to $A_q(\Delta)$, and by (26)

$$h(z) = \ell(\bigcap_q \Omega) , \quad \tilde{h}(z) = (1 - ze^{-i\theta})^{q-2} \ell(\bigcap_q \tilde{\Omega}) ,$$

so that by (27)

$$(32) \quad h(z) = \tilde{h}(z) = 0 \quad \text{for} \quad |z| \geq 1 .$$

From well known properties of logarithmic potentials we conclude that h and \tilde{h} are continuous everywhere and that, in view of the second equation (32),

$$(33) \quad \left| \iint_{|\zeta| < 1} \frac{(1 - |\zeta|^2)^{q-2} v(\zeta) d\xi d\eta}{(1 - \zeta e^{-i\theta})^{q-2} (\zeta - z)} \right| \leq c (1 - |z|) \log \frac{1}{1 - |z|}$$

for $|z| < 1$, where c does not depend on θ . Also

$$(34) \quad \frac{\partial h}{\partial \bar{z}} = \frac{\partial \tilde{h}}{\partial \bar{z}} = (1 - |z|^2)^{q-2} v(z) \quad \text{for} \quad |z| < 1$$

(in the sense of weak derivatives). By (32) and (34) we have that $h \equiv \tilde{h}$. Noting (33) and the fact that θ was arbitrary we conclude that

$$(35) \quad h(z) = 0(-(1 - |z|)^{q-1} \log(1 - |z|)) , \quad |z| \uparrow 1 .$$

One computes easily from (34) and (25) that for every fixed $A \in G$ the function

$$h(A(z))A'(z)^{1-q} - h(z)$$

is holomorphic in $|z| < 1$. Since it vanishes on $|z| = 1$ we have that

$$(36) \quad h(A(z)) = h(z)A'(z)^{q-1} \quad \text{for } A \in G.$$

Using these properties of h we shall show that $\ell \equiv 0$.

14. Let ω be the closure in Δ of the set

$$\{z \in \Delta \mid |A(z)| > |z| \quad \text{for } \text{id} \neq A \in G\}$$

and let ω_r be the intersection of ω with $|z| < r < 1$. Then ω is a fundamental region. For every r , $0 < r < 1$, the boundary σ_r of ω_r consists of a portion γ_r of the circle $|z| = r$ and of $2n = 2n(r)$ circular arcs $\delta_1, \dots, \delta_n, \delta'_1, \dots, \delta'_n$ such that there exist elements A_1, \dots, A_n of G with

$$(37) \quad A_j(\delta_j) = -\delta'_j, \quad j = 1, \dots, n.$$

All this is known and easy to check.

Now let $\phi \in A_q(\Delta, G)$ be given. By (24) and (34)

$$\begin{aligned} \ell(\phi) &= \lim_{r \uparrow 1} \iint_{\omega_r} (1 - |z|^2)^{q-2} v(z) \phi(z) dx dy \\ &= \lim_{r \uparrow 1} \iint_{\omega_r} \phi \frac{\partial h}{\partial \bar{z}} dx dy = \frac{1}{2} \lim_{r \uparrow 1} \int_{\sigma_r} \phi h dz. \end{aligned}$$

Since by (34) we have that

$$\phi(A(z))h(A(z))A'(z) = \phi(z)h(z) \quad \text{for } A \in G,$$

it follows from (37) that

1. 设 $f(x) = x^2 + 2x + 1$, $g(x) = x^2 - 2x + 1$, 求 $f(x) + g(x)$ 和 $f(x) - g(x)$.

2. 设 $f(x) = x^2 + 3x + 2$, $g(x) = x^2 - 3x + 2$, 求 $f(x) \cdot g(x)$.

3. 设 $f(x) = x^2 + 4x + 4$, $g(x) = x^2 - 4x + 4$, 求 $f(x) - g(x)$.

$$f(x) = x^2 + 2x + 1$$

4. 设 $f(x) = x^2 + 5x + 6$, $g(x) = x^2 - 5x + 6$, 求 $f(x) + g(x)$.

5. 设 $f(x) = x^2 + 6x + 9$, $g(x) = x^2 - 6x + 9$, 求 $f(x) - g(x)$.

6. 设 $f(x) = x^2 + 7x + 12$, $g(x) = x^2 - 7x + 12$, 求 $f(x) \cdot g(x)$.

7. 设 $f(x) = x^2 + 8x + 16$, $g(x) = x^2 - 8x + 16$, 求 $f(x) - g(x)$.

8. 设 $f(x) = x^2 + 9x + 20$, $g(x) = x^2 - 9x + 20$, 求 $f(x) + g(x)$.

9. 设 $f(x) = x^2 + 10x + 25$, $g(x) = x^2 - 10x + 25$, 求 $f(x) - g(x)$.

10. 设 $f(x) = x^2 + 11x + 30$, $g(x) = x^2 - 11x + 30$, 求 $f(x) \cdot g(x)$.

11. 设 $f(x) = x^2 + 12x + 36$, $g(x) = x^2 - 12x + 36$, 求 $f(x) - g(x)$.

12. 设 $f(x) = x^2 + 13x + 42$, $g(x) = x^2 - 13x + 42$, 求 $f(x) + g(x)$.

13. 设 $f(x) = x^2 + 14x + 49$, $g(x) = x^2 - 14x + 49$, 求 $f(x) - g(x)$.

$$\int_{\delta_j} \phi h dz + \int_{\delta'_j} \phi h dz = 0, \quad j = 1, \dots, n,$$

so that

$$-2i\ell(\phi) = \lim_{r \uparrow 1} \int_{\gamma_r} \phi h dz,$$

and by (35)

$$(38) \quad |\ell(\phi)| \leq \text{const.} \liminf_{r \uparrow 1} (1-r) \log \frac{1}{1-r} \int_{\gamma_r} |\phi| |dz|.$$

Since

$$\int_{1/2}^1 (1-r^2) \int_{\gamma_r} |\phi| |dz| dr \leq \|\phi\|_{A_q(\Delta, G)}$$

(38) implies that $\ell(\phi) = 0$. Q.E.D.

15. For $\bar{\Phi} \in A_q(D)$, $\psi \in B_q(D, G)$ we have that

$$(39) \quad (\bar{\Phi}, \psi)_q = (\hat{\kappa}_q \bar{\Phi}, \psi)_{q, G}.$$

Indeed this means that

$$\begin{aligned} \sum_{A \in G} \iint_{A(\omega)} \lambda(z)^{2-2q} \bar{\Phi}(z) \overline{\psi(z)} dx dy \\ = \iint_{\omega} \lambda(z)^{2-2q} \overline{\psi(z)} \sum_{A \in G} \bar{\Phi}(A(z)) A'(z)^q dx dy \end{aligned}$$

which is easily verified.

16. Proof of Theorem 1. Assume that $\psi \in B_q(D, G)$ is such that $(\phi, \psi)_{q, G} = 0$ for all $\phi \in A_q(D, G)$, then $(\otimes_q \bar{\Phi}, \psi)_{q, G} = 0$ for all $\bar{\Phi} \in A_q(D)$ and by (39) also $(\bar{\Phi}, \psi)_q = 0$. Hence $\psi = 0$ by the result in 10.

Now let $\ell(\phi)$ be a given linear functional on $A_q(D, G)$. Then (cf. 12) there is a $v \in L_{\infty}(D)$ satisfying (25) such that (24) holds.

Set $\psi = \beta_q \bar{v}$. Then (cf. 9) $\psi \in B_q(D)$ and by (26) and (23)

$$(40) \quad \ell(\bigotimes_q \Phi) = (\Phi, \psi)_q \quad \text{for} \quad \Phi \in A_q.$$

Now, for $A \in G$ and $B = A^{-1}$

$$\psi(A(z))A'(z)^q = c_q \iint_D A'(z)^q \lambda(\zeta)^{-q} k(z, \zeta)^q \overline{v(\zeta)} d\xi d\eta.$$

Setting $\zeta = B\xi$ and noting (9), (13) we obtain

$$\begin{aligned} \psi(A(z))A'(z)^q &= c_q \iint_D A'(z)^q \lambda(\zeta)^{2-q} k(A(z), \zeta)^q \overline{v(\zeta)} d\xi d\eta \\ &= c_q \iint_D \lambda(A \circ B(\xi))^{2-q} A'(z)^q k(A(z), A \circ B(\xi))^q \overline{v(A \circ B(\xi))} d\xi d\eta \\ &= c_q \iint_D \lambda(B(\xi))^{2-q} k(z, B(\xi))^q \overline{v(B(\xi))} |B'(\xi)|^2 d\xi d\eta = \psi(z). \end{aligned}$$

Thus $\psi \in B_q(D, G)$ and, by (39) and (40),

$$\ell(\phi) = (\phi, \psi)_{q, G}$$

whenever $\phi \in \bigotimes_q A_q(D)$. In view of (29) the same holds for all $\phi \in A_q(D, G)$.

17. Proof of Theorem 2. In view of 11 we must show only that $\bigotimes_q A_q(D) = A_q(D, G)$. Let χ be the characteristic function of a fundamental region ω . Then $\chi \lambda^{2-q} \phi \in L_\omega(D)$ and we may form $\hat{\phi} = \bigotimes_q \alpha_q(\chi \lambda^{2-q} \phi)$ which belongs to $A_q(D, G)$. Let ψ be any element in $B_q(D, G)$. By (39)

1. The first part of the document discusses the importance of maintaining accurate records of all transactions and activities. It emphasizes that this is crucial for ensuring transparency and accountability in the organization's operations.

2. The second part of the document outlines the various methods and tools used to collect and analyze data. It highlights the need for a systematic approach to data collection and the importance of using reliable sources of information.

3. The third part of the document describes the process of identifying and measuring the key performance indicators (KPIs) that are most relevant to the organization's goals. It stresses the importance of setting clear, measurable targets and regularly monitoring progress.

4. The fourth part of the document discusses the challenges and limitations of the data collection and analysis process. It acknowledges that there are many factors that can affect the quality and reliability of the data, and it provides suggestions for how to minimize these risks.

5. The fifth part of the document concludes by summarizing the key findings and recommendations of the study. It reiterates the importance of maintaining accurate records and using a systematic approach to data collection and analysis, and it provides a final set of recommendations for the organization.

6. The sixth part of the document provides a detailed description of the data collection and analysis process, including a list of the specific methods and tools used, and a description of the data sources and the analysis techniques employed.

7. The seventh part of the document discusses the results of the data collection and analysis process, including a summary of the key findings and a detailed description of the data trends and patterns observed.

8. The eighth part of the document provides a detailed description of the data collection and analysis process, including a list of the specific methods and tools used, and a description of the data sources and the analysis techniques employed.

9. The ninth part of the document discusses the results of the data collection and analysis process, including a summary of the key findings and a detailed description of the data trends and patterns observed.

$$\begin{aligned}
(\hat{\phi}, \psi)_{q,G} &= (\alpha_q(\chi \lambda^{2-q}\phi), \psi)_q \\
&= c_q \iint_D \lambda(z)^{2-2q} \overline{\psi(z)} \iint_{\omega} \lambda(\zeta)^{2-2q} \phi(\zeta) k(z, \zeta)^q d\xi d\eta dx dy \\
&= c_q \iint_{\omega} \lambda(\zeta)^{2-2q} \phi(\zeta) \iint_D \lambda(z)^{2-2q} \overline{k(\zeta, z)^q \psi(z)} dx dy d\xi d\eta \\
&= (\phi, \psi)_{q,G} .
\end{aligned}$$

Hence

$$(41) \quad \phi = \bigotimes_q \alpha_q(\chi \lambda^{2-q}\phi) ,$$

by Theorem 1.

18. Proof of Theorem 3. Let χ be as in the previous proof. We shall show that if $\phi \in B_q(D, G)$, then

$$(42) \quad \phi = \bigotimes_q \beta_q(\chi \lambda^{-q}\phi)$$

(note that $\chi \lambda^{-q}\phi \in L_{\infty}(D)$). By (16)

$$\phi(z) = \sum_{A \in G} c_q \iint_{A(\omega)} \lambda(\zeta)^{2-2q} k(z, \zeta)^q \phi(\zeta) d\xi d\eta$$

this series being absolutely and normally convergent. Setting

$B = A^{-1}$ and using (2), (9) and (12) we obtain

$$\begin{aligned}
\phi(z) &= \sum_{A \in G} c_q \iint_{A(\omega)} \lambda(B(\zeta))^{2-2q} k(B(z), B(\zeta))^q \phi(B(\zeta)) B'(z)^q |B'(\zeta)|^2 d\xi d\eta \\
&= \sum_{A \in G} c_q B'(z)^q \iint_{\omega} \lambda(\zeta)^{2-2q} k(B(z), \zeta)^q \phi(\zeta) d\xi d\eta
\end{aligned}$$

which is precisely (42).

§4. Periods of automorphic forms

19. Let $D = U$ so that G is a group of Möbius transformations $z \rightarrow A(z) = (az+b)/(cz+d)$. Let \prod_{2q-2} denote the additive groups of polynomials $P(z) = \sum_{j=0}^{2q-2} \alpha_j z^j$. The group G operates from the right on \prod_{2q-2} by the rule

$$(43) \quad (PA)(z) = P(A(z))A'(z)^{1-q}.$$

A mapping $A \rightarrow P_A$ of G into \prod_{2q-2} is called a cocycle of

$$(44) \quad P_{AB} = P_A B + P_B,$$

a coboundary if there exists an element $Q \in \prod_{2q-2}$ such that

$$(45) \quad P_A = QA - Q.$$

The coboundaries form a subgroup of the additive group of cocycles. The factor group (cocycles/coboundaries) is denoted by $H^1(G, \prod_{2q-2})$.

20. Let ϕ be an automorphic form of weight $(-2q)$ and F a holomorphic function such that

$$(46) \quad \frac{d^{2q-1} F(z)}{dz^{2q-1}} = \phi(z).$$

One verifies easily that for every $A \in G$ the $(2q-1)$ -st derivative of

$$(47) \quad F(A(z))A'(z)^{1-q} - F(z)$$

vanishes, so that this function belongs to \prod_{2q-2} . We call it the Eichler period of F on A . The mapping

... ..

... ..

... ..

... ..

... ..

... ..

... ..

... ..

... ..

... ..

... ..

... ..

... ..

... ..

... ..

... ..

... ..

... ..

... ..

... ..

... ..

... ..

... ..

... ..

... ..

$$(48) \quad A \longrightarrow F(A(z))A'(z)^{1-q} - F(z)$$

is clearly a cocycle. Since F is determined by ϕ modulo a polynomial of degree at most $2q-2$, the cohomology class of (48) depends only on ϕ and depends on ϕ linearly. We call it the Eichler class of ϕ .

The existence of an F satisfying (46) and the condition

$$(49) \quad F(A(z))A'(z)^{1-q} = F(z) \quad \text{for all } A \in G$$

is necessary and sufficient for the vanishing of the Eichler class of ϕ .

21. Let $\phi \in B_q(U, G)$. Then $|\phi(x+iy)| \leq \|\phi\|_{B_q(U, G)} y^{-q}$ so that every $F(z)$ satisfying (46) is continuous on the real axis. Assume that (49) holds and let $x \in \mathbb{R}$ be a fixed point of a hyperbolic parabola element A of G . Then $A(x) = x$, $A'(x) \neq 1$ and, by (49), $F(x) = 0$. Hence also

$$(50) \quad F(x) = 0 \quad \text{for } x \in \bigwedge(G)$$

where $\bigwedge(G)$ is the closure of the set of fixed points. Conversely, if (46) and (50) hold, then for every fixed $A \in G$ the polynomial (47) vanishes on $\bigwedge(G)$ since $A(\bigwedge(G)) = \bigwedge(G)$ for every $A \in G$. If G is not elementary, $\bigwedge(G)$ is infinite and we conclude that (49) holds.

22. Proof of Theorem 4. If G is of the first kind and the Eichler class of $\phi \in B_q(U, G)$ is zero, then $\phi(z) = F^{(2q-1)}(z)$ with $F = 0$ on \mathbb{R} . Hence $F \equiv 0$, $\phi \equiv 0$.

23. Let \bigwedge be a perfect set on the real axis (in the next paragraph we shall take $\bigwedge = \bigwedge(G)$ for a non-elementary group G of

Introduction

The purpose of this document is to provide a comprehensive overview of the project's objectives, scope, and the methodology used to achieve the desired outcomes. This document serves as a guide for the project team and stakeholders, ensuring that everyone is aligned with the project's goals and the approach taken to achieve them.

The project is designed to address the challenges faced by the organization in the current market environment. By implementing the proposed solutions, we aim to improve operational efficiency, reduce costs, and enhance the overall quality of our services. The project's success will be measured by the achievement of these key performance indicators (KPIs).

The methodology adopted for this project is a structured approach that involves several key phases:
1. **Requirement Gathering:** Understanding the needs and expectations of the stakeholders.
2. **Analysis:** Identifying the root causes of the problems and determining the most effective solutions.
3. **Design:** Developing a detailed plan for the implementation of the solutions.
4. **Implementation:** Executing the plan and monitoring the progress.
5. **Evaluation:** Assessing the results and making necessary adjustments.

The project team consists of experts in various fields, including project management, business analysis, and technical implementation. We have a strong track record of successfully completing similar projects, and we are confident that we will achieve the same level of success in this project. The project is supported by senior management, ensuring that we have the necessary resources and authority to move forward.

We anticipate that the project will be completed within the specified timeline and budget. The final deliverables will be presented to the stakeholders for their review and approval. We will continue to maintain open communication throughout the project, providing regular updates on the progress and any challenges encountered. Our goal is to ensure that the project meets the highest standards of quality and that it delivers the expected benefits to the organization.

the second kind). Let a_1, \dots, a_q be distinct points of Λ and set

$$(51) \quad p(z) = (z - a_1)(z - a_2) \dots (z - a_q) .$$

Then every rational function with simple poles in Λ which belongs to $A_q(U)$ is of the form

$$(52) \quad \sum_{j=1}^n \frac{\alpha_j}{(z - x_j)p(z)}$$

where x_1, \dots, x_n are distinct points of Λ and $x_j \neq a_k$ and the α_j are arbitrary complex constants. Indeed, a rational function with no singularities except perhaps for simple poles at $a_1, \dots, a_q, x_1, \dots, x_n$ belongs to $A_q(U)$ if and only if it is of the form

$$\sum_{j=1}^q \frac{\beta_j}{z - a_j} + \sum_{j=1}^n \frac{\gamma_j}{z - x_j}$$

with

$$\sum_{j=1}^q \beta_j a_j^s + \sum_{j=1}^n \gamma_j x_j^s = 0 , \quad s = 0, 1, \dots, q-1 .$$

The space of such functions has therefore dimension n . On the other hand (52) always belongs to $A_q(U)$.

If $\Phi(z) \in A_q(U)$ is a rational function with poles in Λ it is a limit of functions of the form (52). Indeed, if ξ_1, \dots, ξ_m are the poles of Φ and v_1, \dots, v_m their multiplicities we have $0 < v_j \leq q-1$ and

$$\Phi(z) = r(z) \prod_{j=1}^m (z - \xi_j)^{-v_j}$$

where $r(z)$ is a polynomial of degree at most $v_1 + \dots + v_m + q-1$ with $r(\xi_j) \neq 0$. Let $\varepsilon > 0$ be given. Since Λ is perfect there exist

distinct points ξ_{jk} , $j = 1, \dots, m$, $k = 1, \dots, v_j$ in Λ with $|\xi_{jk} - \xi_j| < \varepsilon$. The function

$$\tilde{\Phi}(z) = r(z) \prod_{j=1}^m \prod_{k=1}^{v_j} (z - \xi_{jk})^{-1}$$

is of the form (52) and one verifies that $\|\Phi - \tilde{\Phi}\|_{A_q(U)}$ will be arbitrarily small for ε sufficiently small.

24. Proof of Theorem 5. Let $\Lambda = \Lambda(G)$ and let a_1, \dots, a_q and $p(z)$ be as in 23. Let $\phi \in B_q(U, G)$. Noting (11), (12) we write (16) in the form

$$\phi(z) = \frac{(-1)^q (2q-1)}{\pi} \iint_{\eta > 0} \frac{|\zeta - \bar{\zeta}|^{2q-2} \phi(\zeta) d\xi d\eta}{(\bar{\zeta} - z)^{2q}}.$$

Set

$$G(z) = \iint_{\eta > 0} \frac{|\zeta - \bar{\zeta}|^{2q-2} \phi(\zeta) d\xi d\eta}{(\bar{\zeta} - z)p(\bar{\zeta})}.$$

This function is holomorphic in U and continuous everywhere except perhaps at the points a_j . Next, set

$$F(z) = \frac{(-1)^q p(z) G(z)}{\pi (2q-2)!}.$$

Then $F(a_j) = 0$, $j = 1, \dots, q$ and since

$$\frac{p(z)}{p(\bar{\zeta})(\bar{\zeta} - z)} - \frac{1}{\bar{\zeta} - z}$$

is a polynomial of degree $q-1$ in z , we have that

$$F^{(2q-1)}(z) = \frac{(-1)^q (2q-1)}{\pi} \iint_{\eta > 0} \frac{|\zeta - \bar{\zeta}|^{2q-2} \phi(\zeta) d\xi d\eta}{(\bar{\zeta} - z)^{2q}}$$



in U , so that (46) holds. By 21 the Eichler class of ϕ vanishes if and only if $F(x) = 0$ for $x \in \Lambda(G)$, $x \neq a_j$. This condition is equivalent to

$$G(x) = 0 \quad \text{on} \quad \Lambda(G) - \{a_1, \dots, a_q\}.$$

But for a real $x \neq a_j$

$$\pi \overline{G(x)} = (-1)^q (2q-1) (\overline{\Phi}, \phi)_q$$

where

$$\overline{\Phi}(x) = \frac{1}{(z-x)p(z)} \in A_q(U).$$

The conclusion of Theorem 5 now follows from 23.

25. Let G be again a Fuchsian non-elementary group of the second kind and let Ω denote the complement of $\Lambda(G)$ in the extended complex plane. Then there exists a Fuchsian group H_0 without elliptic elements and a holomorphic mapping $\zeta \rightarrow g(\zeta)$ of U onto Ω such that if $\zeta_1, \zeta_2 \in U$, then $g(\zeta_1) = g(\zeta_2)$ if and only if there is a $C \in H_0$ with $C(\zeta_1) = \zeta_2$. Also, there is a Fuchsian group H such that if $\zeta_1, \zeta_2 \in U$, then $A(g(\zeta_1)) = g(\zeta_2)$ for some $A \in G$ if and only if there is a $B \in H$ with $B(\zeta_1) = \zeta_2$. The mapping τ of H onto G which sends $B \in H$ into $A \in G$ with $g \circ B = A \circ g$ is a holomorphism; its kernel is precisely H_0 .

Let $\phi \in A_2^f(U, G)$. This means that $\phi \in A_2(U, G)$ and $\phi(z)$ is holomorphic in Ω and satisfies the relation

$$(53) \quad \phi(\bar{z}) = \overline{\phi(z)}.$$

Let ω be a fundamental region for G in Ω chosen so that $\omega \cap U$ is simply connected and ω is invariant under the mapping $z \rightarrow \bar{z}$.

Then there is a fundamental region $\hat{\omega}$ for H in U such that $g(\hat{\omega}) = \omega$. Let $K \subset H$ contain exactly one representative of each coset of H modulo H_0 . Then

$$\hat{\omega}_0 = \bigcup_{B \in K} B(\hat{\omega})$$

is a fundamental region for H_0 in U and $g(\hat{\omega}_0) = \Omega$.

Set $\hat{\phi}(\zeta) = \phi(g(\zeta))g'(\zeta)^2$. Then

$$\iint_{\omega} |\hat{\phi}(\zeta)| d\xi d\eta = \iint_{\omega} |\phi(z)| dx dy = 2 \|\phi\|_{A_2(U, G)}$$

by (53), and for $B \in H$ we have that

$$\begin{aligned} \hat{\phi}(B(\zeta))B'(\zeta)^2 &= \phi(g(B(\zeta)))g'(B(\zeta))^2B'(\zeta)^2 \\ &= \phi(A(g(\zeta))A'(g(\zeta))^2g'(\zeta)^2 = \phi(g(\zeta))g'(\zeta)^2 = \hat{\phi}(\zeta) \end{aligned}$$

where A is the image of B under the homomorphism τ described above.

Hence $\hat{\phi} \in A_2(U, H)$ and by Theorem 2 we have that $\hat{\phi} = \bigotimes_{2, H} \hat{\Phi}$, $\hat{\Phi} \in A_2(U)$, or

$$\begin{aligned} (54) \quad \hat{\phi}(\zeta) &= \sum \hat{\Phi}(B(\zeta))B'(\zeta)^2 \\ &= \sum_{B \in K} \sum_{C \in H_0} \hat{\Phi}(C(B(\zeta))C'(B(\zeta))^2B'(\zeta)^2. \end{aligned}$$

Set

$$\hat{\Phi}_0(\zeta) = \sum_{C \in H_0} \hat{\Phi}(C(\zeta))C'(\zeta)^2.$$

Then $\hat{\Phi}_0 = \bigotimes_{2, H_0} \hat{\Phi} \in A_2(U, H_0)$. Hence there exists a holomorphic function $\overline{\Phi}_0(z)$, $z \in \Omega$ such that $\hat{\Phi}_0(\zeta) = \overline{\Phi}_0(g(\zeta))g'(\zeta)^2$; we have that

1. $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) = \int_0^1 f(x) dx$
 2. $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) = \int_0^1 f(x) dx$
 3. $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) = \int_0^1 f(x) dx$

4. $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) = \int_0^1 f(x) dx$
 5. $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) = \int_0^1 f(x) dx$

6. $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) = \int_0^1 f(x) dx$
 7. $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) = \int_0^1 f(x) dx$

8. $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) = \int_0^1 f(x) dx$

9. $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) = \int_0^1 f(x) dx$

10. $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) = \int_0^1 f(x) dx$

11. $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) = \int_0^1 f(x) dx$

12. $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) = \int_0^1 f(x) dx$
 13. $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) = \int_0^1 f(x) dx$

14. $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) = \int_0^1 f(x) dx$

15. $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) = \int_0^1 f(x) dx$

16.

17. $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) = \int_0^1 f(x) dx$

18. $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) = \int_0^1 f(x) dx$

19. $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) = \int_0^1 f(x) dx$
 20. $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) = \int_0^1 f(x) dx$

$$\iint_{\Omega} |\Phi_0(z)| dx dy < \infty$$

since this integral equals $\|\hat{\Phi}_0\|_{A_2(U, H_0)}$.

Now (54) may be written as

$$\begin{aligned} \phi(g(\zeta))g'(\zeta)^2 &= \sum_{B \in K} \hat{\Phi}_0(B(\zeta))B'(\zeta)^2 \\ &= \sum_{B \in K} \Phi_0(g(B(\zeta)))g'(B(\zeta))^2 B'(\zeta)^2 \\ &= \sum_{A \in G} \Phi_0(A(g(\zeta)))A'(g(\zeta))^2 g'(\zeta)^2 \end{aligned}$$

Thus every $\phi \in A_2^\#(U, G)$ admits the representation

$$(55) \quad \phi = \bigoplus_{2, G} \Phi_0$$

where Φ_0 is holomorphic in Ω and absolutely integrable over this domain.

26. Proof of Theorem 6. Assume that $\psi \in B_2(U, G)$ is orthogonal to $A_2^\#(U, G)$. Let $r(z)$ be a rational function with poles in $\Lambda(G)$ belonging to $A_2(U)$ and $\phi = \bigoplus_{2, G} r$. Since

$$\iint_{\Omega} |r(z)| dx dy < \infty$$

(Ω having the same meaning as in 25) the argument in 11 can be repeated to show that the Poincaré series

$$\sum_{A \in G} r(A(z))A'(z)^2$$

converges absolutely and normally in Ω . This implies that

$\phi(z) = \phi_1(z) + i\phi_2(z)$, with $\phi_1, \phi_2 \in A_2^\#(U, G)$. Hence $(\phi, \psi)_{2, G} = 0$.

By Theorem 5 the Eichler class of ψ is zero.

Assume next that the Eichler class of $\psi \in B_2(U, G)$ vanishes and let $\phi \in A_2^\#(U, G)$. Then ϕ admits the representation (55). By the approximation theorem proved in [2] there exists a sequence of rational functions $\{r_j(z)\}$ with poles in $\Lambda(G)$ such that

$$(56) \quad \iint_{\Omega} |r_j(z) - \Phi_0(z)| dx dy \rightarrow 0.$$

By Theorem 5 we have that $(\mathbb{H})_{2, G}^{r_j, \psi} = 0$. Since (56) implies that $\|r_j - \Phi_0\|_{A_2(U)} \rightarrow 0$, we have that $(\mathbb{H})_{2, G}^{r_j} \rightarrow \phi$ in $A_2(U, G)$, by Theorem 2. Therefore $(\phi, \psi)_{2, G} = 0$.

85. Finitely generated Fuchsian groups

27. A Riemann surface S will be called of finite type, more precisely of type (g, n, m) , if it is conformally equivalent to $S_0 - \sigma$ where S_0 is a closed (compact) surface of genus g and σ a closed set with $n+m \geq 0$ components of which $n \geq 0$ are points and $m \geq 0$ simply connected non-degenerate continua. The numbers g, n, m depend only on S ; we say that S has n punctures and m boundary curves.

If $m = 0$ then S_0 (the natural compactification of S) is determined by S except for conformal equivalence. If $m > 0$ there exists a Riemann surface S_1 of type $(2g+m-1, 2n)$ (the double of S) which is determined by S except for conformal equivalence, m disjoint simple closed analytic curves $\gamma_1, \dots, \gamma_m$ on S_1 and an anti-conformal involution ρ of S_1 which leaves a point $p \in S_1$ fixed if and only if $p \in \gamma = \gamma_1 \cup \dots \cup \gamma_m$, such that $S_1 - \gamma$ consists of two components one of which is conformally equivalent to S .

The first part of the proof is devoted to showing that the
 function f is continuous at x_0 . Let $\epsilon > 0$ be given. Then
 there exists a $\delta > 0$ such that for all x with $|x - x_0| < \delta$,
 we have $|f(x) - f(x_0)| < \epsilon$.

$$f(x) = \lim_{n \rightarrow \infty} \left(\frac{1}{n} \sum_{k=1}^n f(x_k) \right) \quad (1)$$

Now, let x_0 be a point of discontinuity of f . Then there exists a
 sequence $\{x_n\}$ such that $x_n \rightarrow x_0$ but $f(x_n) \not\rightarrow f(x_0)$.
 Let $\epsilon = \frac{1}{2} |f(x_0) - f(x_n)|$. Then for all n , we have
 $|f(x_n) - f(x_0)| > 2\epsilon$.

Let $\delta > 0$ be such that for all x with $|x - x_0| < \delta$,
 we have $|f(x) - f(x_0)| < \epsilon$. Then for all n , we have
 $|f(x_n) - f(x_0)| > 2\epsilon$. This contradicts the fact that
 $x_n \rightarrow x_0$ and $f(x_n) \rightarrow f(x_0)$. Therefore, f is
 continuous at x_0 .

The second part of the proof is devoted to showing that the
 function f is differentiable at x_0 . Let $h > 0$ be given. Then
 there exists a $\delta > 0$ such that for all h with $|h| < \delta$,
 we have $|f(x_0 + h) - f(x_0) - f'(x_0)h| < \epsilon$.
 Let $\epsilon > 0$ be given. Then there exists a $\delta > 0$ such that
 for all h with $|h| < \delta$, we have $|f(x_0 + h) - f(x_0) - f'(x_0)h| < \epsilon$.
 Let $h > 0$ be given. Then there exists a $\delta > 0$ such that
 for all h with $|h| < \delta$, we have $|f(x_0 + h) - f(x_0) - f'(x_0)h| < \epsilon$.

The fundamental group $\pi_1(S)$ is finitely generated if and only if S is of finite type. This is a known result in surface topology.

28. Let D_G/G be of finite type. Then G is finitely generated.

This is well known and can be proved by dissecting D_G/G by finitely many smooth curves into a simply connected region such that a component of its inverse image under the projection $D_G \rightarrow D_G/G$ is a fundamental domain whose boundary consists of finitely many "sides".

29. Let S be a Riemann surface. An Abelian differential (of the first kind) on S is a rule associating with every local $p \rightarrow t(p)$ defined on a domain $G \subset S$ a holomorphic function $\phi(t)$ such that $\phi(t)dt$ is invariant under parameter changes. In this case $|\phi(t)|^2$ is a density. If we demand instead the invariance of $\phi(t)dt^2$ we obtain a quadratic (holomorphic) differential; now $|\phi(t)|$ is a density. The Abelian differentials α with

$$\iint_S |\alpha|^2 < \infty$$

form a Hilbert space $A_1(S)$ of square integrable differentials. The quadratic differentials β with

$$(57) \quad \iint_S |\beta| < \infty$$

form the Banach space $A_2(S)$ of integrable differentials. We have that

$$\dim A_1(S) \leq \dim A_2(S)$$

The first part of the paper is devoted to the study of the
 properties of the function $f(x)$ defined by the equation

$$f(x) = \int_0^x \frac{1}{1+t^2} dt$$

It is shown that $f(x)$ is a continuous function of x and
 that it satisfies the functional equation

$$f(x) + f\left(\frac{1}{x}\right) = \frac{\pi}{2}$$
 for all $x \neq 0$. This equation is satisfied by the function
 $f(x) = \arctan x$ and it is proved that $f(x)$ is the only
 continuous function satisfying this equation.

In the second part of the paper the properties of the
 function $f(x)$ are studied for $x > 0$. It is shown that
 $f(x)$ is a strictly increasing function of x and that
 $f(x) < \frac{\pi}{2}$ for all $x > 0$. It is also shown that
 $f(x)$ is a concave down function of x and that
 $f(x) > \frac{\pi}{4}$ for all $x > 1$. Finally, it is shown that
 $f(x)$ is a function of bounded variation on the interval
 $[0, \infty)$ and that its total variation is $\frac{\pi}{2}$.

The third part of the paper is devoted to the study of the
 function $f(x)$ for $x < 0$. It is shown that $f(x)$ is a
 strictly decreasing function of x and that $f(x) > \frac{\pi}{2}$

$$\text{for all } x < 0. \quad (2)$$

In the fourth part of the paper the properties of the
 function $f(x)$ are studied for $x = 0$. It is shown that
 $f(0) = 0$ and that $f(x)$ is a function of bounded variation
 on the interval $[-\infty, 0]$. Finally, it is shown that the
 total variation of $f(x)$ on the interval $[-\infty, 0]$ is $\frac{\pi}{2}$.

because if $\alpha_1, \alpha_2 \in A_1(S)$, then $\alpha_1 \alpha_2 \in A_2(S)$. If the genus of S is infinite, then $\dim A_1(S) = \infty$ (cf. Nevanlinna [4]) and hence $\dim A_2(S) = \infty$. If the genus of S is $g < \infty$, then $S = S_0 - \sigma$ where σ is a closed set on the closed surface S_0 of genus g . If S contains N distinct points, then $\dim A_2(S) \geq N$ since it is known (say from the Riemann-Roch theorem) that to every $p \in S_0$ there is a meromorphic quadratic differential β_p on S_0 whose only singularity is a simple pole at p . We conclude that

$$(58) \quad \dim A_2(S) = \infty \quad \text{unless } S \text{ is of finite type } (g, n, 0) .$$

30. The space $A_2(D, G)$ can be defined even when G is a discrete group of conformal self-mappings of a non-simply connected domain (since λ does not enter in the definition of this space). Let D_G denote D with the fixed points of elements of G (distinct from the identity) removed. Then there is a canonical isomorphism

$$(59) \quad A_2(D, G) \cong A_2(D_G/G) .$$

Indeed, $A_2(D, G)$ may be identified with the space X of meromorphic quadratic differentials β on the Riemann surface D/G for which (57) holds and which have no singularities except perhaps simple poles on the set σ consisting of the images of fixed points of G under the projection $D \rightarrow D/G$. Since σ is discrete and $D/G - \sigma = D_G/G$, X may be identified with $A_2(D_G/G)$.

31. Let G be a Fuchsian group. The elements of $B_q(U, G)$ with vanishing Eichler class form a closed linear subspace $B_q^0(U, G)$.

If G is finitely generated, $\dim B_q(U, G)/B_q^0(U, G) < \infty$.

The first part of the paper is devoted to the study of the asymptotic behavior of the solutions of the system of equations (1) as $t \rightarrow \infty$. It is shown that the solutions of this system tend to zero as $t \rightarrow \infty$ if and only if the matrix A is positive definite. In the second part of the paper, the problem of the stability of the solutions of the system (1) is considered. It is shown that the solutions of this system are stable if and only if the matrix A is positive definite. In the third part of the paper, the problem of the asymptotic stability of the solutions of the system (1) is considered. It is shown that the solutions of this system are asymptotically stable if and only if the matrix A is positive definite.

In the fourth part of the paper, the problem of the asymptotic stability of the solutions of the system (1) is considered. It is shown that the solutions of this system are asymptotically stable if and only if the matrix A is positive definite. In the fifth part of the paper, the problem of the asymptotic stability of the solutions of the system (1) is considered. It is shown that the solutions of this system are asymptotically stable if and only if the matrix A is positive definite. In the sixth part of the paper, the problem of the asymptotic stability of the solutions of the system (1) is considered. It is shown that the solutions of this system are asymptotically stable if and only if the matrix A is positive definite.

In the seventh part of the paper, the problem of the asymptotic stability of the solutions of the system (1) is considered. It is shown that the solutions of this system are asymptotically stable if and only if the matrix A is positive definite. In the eighth part of the paper, the problem of the asymptotic stability of the solutions of the system (1) is considered. It is shown that the solutions of this system are asymptotically stable if and only if the matrix A is positive definite. In the ninth part of the paper, the problem of the asymptotic stability of the solutions of the system (1) is considered. It is shown that the solutions of this system are asymptotically stable if and only if the matrix A is positive definite. In the tenth part of the paper, the problem of the asymptotic stability of the solutions of the system (1) is considered. It is shown that the solutions of this system are asymptotically stable if and only if the matrix A is positive definite.

Indeed, assign to every $\phi \in B_q(U, G)$ a holomorphic function $F(z)$, $z \in U$ such that $F^{(2q-1)}(z) = \phi(z)$ and $F^{(v)}(z) = 0$, $v = 0, 1, \dots, 2q-2$. Then $\phi \in B_q^0(U, G)$ whenever the Eichler periods of F vanish on a set of generators of G . This amounts to finitely many linear conditions.

32. Proof of Theorem 7. We may assume that $D = U$. We may assume that G is non-elementary, the theorem being trivial for elementary groups. In view of 27, 28 it suffices to assume that G is finitely generated and to prove that U_G/G is of finite type.

Let G be of the first kind. Then $B_2^0(U, G) = \{0\}$ by Theorem 4, hence $\dim B_2(U, G) < \infty$ by 31, hence $\dim A_2(U, G) < \infty$ by Theorem 1, hence $\dim A_2(U_G/G) < \infty$ by (59), hence U_G/G is of the finite type $(g, n, 0)$ by (58).

Assume next that G is of the second kind. Let $A_2^b(U, G)$ denote the subspace of $A_2(U, G)$ consisting of elements of the form $\phi_1 + i\phi_2$ with $\phi_1, \phi_2 \in A_2^{\#}(U, G)$. By Theorems 1 and 6 the dual space to $A_2^b(U, G)$ is anti-isomorphic to $B_2(U, G)/B_2^0(U, G)$. Thus $\dim A_2^b(U, G) < \infty$. Let Ω have the same meaning as in 25. One sees at once that $A_2^b(U, G)$ may be identified with $A_2(\Omega, G)$. Hence $\dim A_2(\Omega_G/G) < \infty$ by (59), and in view of (58) the Riemann surface $S_1 = \Omega_G/G$ is of finite type $(g, n, 0)$. The mapping $z \rightarrow \bar{z}$ induces an anti-conformal involution ρ on S_1 . The set γ of fixed points of ρ is the image of the intersection of Ω_G with the extended real axis under the canonical mapping $\Omega_G \rightarrow S_1$ and one of the two components of $S_1 - \gamma$ is U_G/G . Hence U_G/G is of finite type (g, n, m) with $n > 0$.

... ..

... ..

... ..

... ..

... ..

... ..

References

1. L. V. Ahlfors, Finitely generated Kleinian groups. To appear.
2. L. Bers, Completeness theorems for Poincare series in one variable. Proc. Int. Symp. on Linear Spaces. Jerusalem Acad. Press - Pergamon Press, 1961, pp. 88-100.
3. L. Bers, An approximation theorem. To appear.
4. R. Nevanlinna, Uniformisierung. Springer, 1953.
5. M. Eichler, Eine Verallgemeinerung der Abelschen Integrale. Math. Z., , pp.
6. A. Selberg, Automorphic functions and integral operators. Seminars on Anal. Functions, Inst. Adv. Study, 1957, vol. II, pp. 152-161.

NYU
IMM-
318

c.1

Bers

At NYU
IMM-
NYU
IMM-
318

c.1
c.1

Bers

Automorphic forms and
Poincaré series...

**N.Y.U. Courant Institute of
Mathematical Sciences**
4 Washington Place
New York 3, N. Y.

